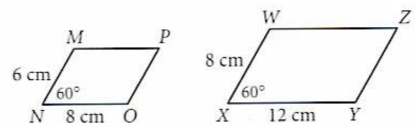
Now look at quadrilaterals *SQUE* and *RHOM*. These figures have corresponding sides that are proportional, but they are *not* similar because the corresponding angles are *not* congruent.

These examples illustrate that for two figures to be similar, both conditions—proportional sides and congruent angles—must hold. Here is another example.

EXAMPLE Determine whether parallelogram *MNOP* is similar to parallelogram *WXYZ*.



▶ Solution $m\angle N = m\angle X$. Using angle properties of parallelograms, $m\angle M = m\angle W = 120°$, $m\angle P = m\angle Z = 60°$, and $m\angle O = m\angle Y = 120°$. So, the corresponding angles are congruent.

However, because $\frac{MN}{WX} = \frac{6}{8} = \frac{3}{4}$ and $\frac{NO}{XY} = \frac{8}{12} = \frac{2}{3}$, the corresponding sides are not proportional. Therefore, the parallelograms are not similar.

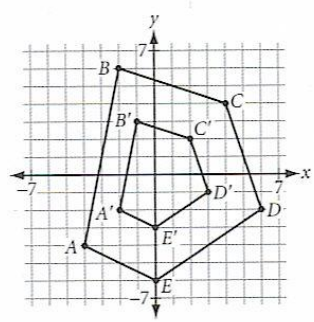
If you know that two polygons are similar, you can use the definition of similar polygons to find missing measures. The example in your book shows you how. Read this example carefully and make sure you understand it.

In Chapter 7, you saw that rigid transformations—translations, rotations, and reflections—preserve the size and shape of a figure, resulting in an image that is congruent to the original. In the next investigation you will look at a nonrigid transformation called a **dilation.**

Investigation 2: Dilations on the Coordinate Plane

To *dilate* a figure in the coordinate plane about the origin, you multiply the coordinates of all its vertices by the same number, called a **scale factor.**

Pentagon *ABCDE* in your book has vertices with coordinates $A(-4, -4)$, $B(-2, 6)$, $C(4, 4)$, $D(6, -2)$, $E(0, -6)$. If you multiply each coordinate by $\frac{1}{2}$, you get $A'(-2, -2)$, $B'(-1, 3)$, $C'(2, 2)$, $D'(3, -1)$, $E'(0, -3)$. The figure at right shows the original pentagon and the image.



If you compare the corresponding sides and angles, you will find that the corresponding angles are congruent and that each side of the image pentagon is half the length of its corresponding original side. So, the pentagons are similar.

Choose one of the other scale factors listed in your book—$\frac{3}{4}$, 2, or 3—and multiply each coordinate of *ABCDE* by the factor you choose. Graph the image pentagon on the same axes as the original pentagon. Compare the corresponding angle measures and side lengths. Are the two pentagons similar? State your findings as a conjecture.

Dilation Similarity Conjecture If one polygon is a dilated image of another polygon, then ________________. **C-90**

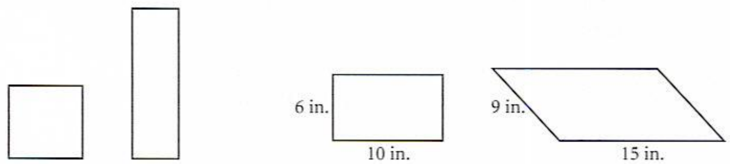
CONDENSED LESSON

11.2 Similar Triangles

In this lesson you will

- Learn **shortcuts** for determining whether two triangles are similar

In Lesson 11.1, you saw that to determine whether two quadrilaterals are congruent, you must check *both* that their corresponding sides are proportional *and* that their corresponding angles are congruent.



Angles are congruent, but quadrilaterals are *not* similar.

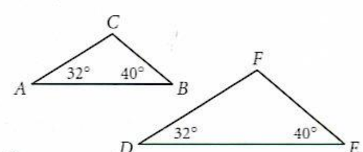
Sides are proportional, but quadrilaterals are *not* similar.

However, triangles are different. In Chapter 4, you discovered that you don't have to check every pair of sides and angles to determine whether two triangles are congruent. You found that SSS, SAS, ASA, and SAA are congruence shortcuts. In this lesson you will find that there are also similarity shortcuts.

Page 589 of your book shows two triangles in which only one pair of angles is congruent. The triangles are clearly not similar. So, knowing only that one pair of angles is congruent is not enough to conclude that two triangles are similar. What if two pairs of angles are congruent?

Investigation 1: Is AA a Similarity Shortcut?

In the triangles at right, $\angle A \cong \angle D$ and $\angle B \cong \angle E$. What must be true about $\angle C$ and $\angle F$? Why?



Measure the sides and compare the ratios of corresponding side lengths. Is $\frac{AB}{DE} \approx \frac{AC}{DF} \approx \frac{BC}{EF}$?

Now draw your own triangle ABC. Use a compass and straightedge to construct triangle DEF, with $\angle A \cong \angle D$ and $\angle B \cong \angle E$. Are your triangles similar? Explain.

Your findings should support this conjecture.

AA Similarity Conjecture If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar. **C-91**

Now consider similarity shortcuts that compare only corresponding sides. The illustration on page 590 of your book shows that you cannot conclude that two triangles are similar given that two pairs of corresponding sides are proportional. What if all three pairs of corresponding sides are proportional?

(continued)

Investigation 2: Is SSS a Similarity Shortcut?

Use the segments on the left to construct $\triangle ABC$. Each segment on the right is three times the length of the corresponding segment on the left. Use the segments on the right to construct $\triangle DEF$.

The side lengths of $\triangle DEF$ are proportional to the side lengths of $\triangle ABC$. Measure the angles and see how they compare. Are the triangles similar?

Construct another pair of triangles so that the side lengths of one triangle are a multiple of the side lengths of the other. Compare the corresponding angles of your triangles.

You can now complete this conjecture.

SSS Similarity Conjecture If the three sides of one triangle are proportional to the three sides of another triangle, then the two triangles are ________________. **C-92**

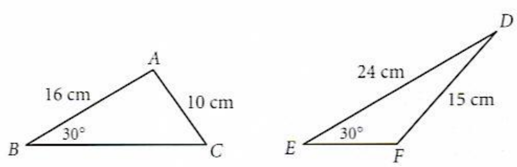
If AA is a similarity shortcut, then so are ASA, SAA, and AAA, because each of those shortcuts contains two angles. That leaves SAS and SSA as possible shortcuts. In the next investigation you will look at SAS.

Investigation 3: Is SAS a Similarity Shortcut?

Try to construct two different triangles that are *not* similar, but that have two pairs of sides proportional and the pair of included angles equal in measure. Can you do it? Your findings should support this conjecture.

SAS Similarity Conjecture If two sides of one triangle are proportional to two sides of another triangle and the included angles are congruent, then the triangles are similar. **C-93**

In the triangles below, two pairs of corresponding sides are proportional and one pair of non-included angles is congruent. However, the triangles are clearly not similar. This shows that SSA is *not* a similarity shortcut.



$$\frac{AB}{DE} = \frac{16}{24} = \frac{2}{3}$$

$$\frac{AC}{DF} = \frac{10}{15} = \frac{2}{3}$$

$\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle B \cong \angle E$, but the triangles are not similar.

CONDENSED LESSON 11.3

Indirect Measurement with Similar Triangles

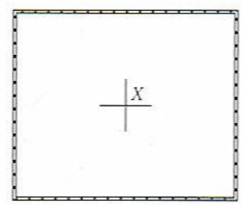
In this lesson you will

- Learn how to use similar triangles to measure tall objects and large distances **indirectly**

Suppose you want to find the height of a tall object such as a flagpole. It would be difficult to measure the flagpole directly—you would need a very tall ladder and a very long tape measure! In this lesson you will learn how you can use similar triangles to find the heights of tall objects indirectly.

Investigation: Mirror, Mirror

You will need another person to help you with this investigation. You will also need a small mirror, a meterstick, and masking tape.



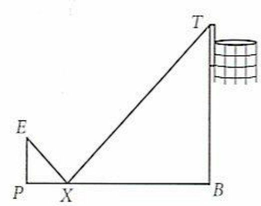
Mark crosshairs on your mirror with masking tape or a soluble pen. Label the intersection point *X*.

Choose a tall object, such as a flagpole, a tall tree, a basketball hoop, or a tall building.

Set the mirror, faceup, on the ground several yards from the object you wish to measure.

Step back from the mirror, staying in line with the mirror and the object, until you see the reflection of the top of the object at point *X* on the mirror.

Have another person measure and record the distances from you to *X* and from *X* to the base of the object. Also, have the person measure and record your height at eye level.



Sketch a diagram of your setup, like this one. Label the top of the object *T*, the base of the object *B*, the point where you stood *P*, and your eye level *E*. Label $\overline{PX}$, $\overline{BX}$, and $\overline{EP}$ with the measurements your helper found.

Think of $\overline{TX}$ as the path of a light ray that bounced back to your eye along $\overline{XE}$. Because the incoming angle must be congruent to the outgoing angle, $\angle EXP \cong \angle TXB$. Also, because $\overline{EP}$ and $\overline{TB}$ are perpendicular to the ground, $\angle P \cong \angle B$. By the AA Similarity Conjecture, $\triangle EPX \sim \triangle TBX$.

Because the triangles are similar, you can set up a proportion to find *TB*, the height of the tall object.

$$\frac{EP}{PX} = \frac{TB}{BX}$$

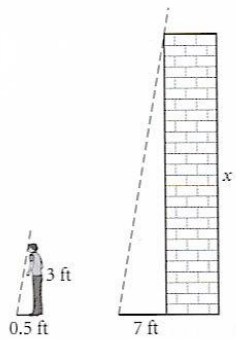
Find the height of your object. Then write a paragraph summarizing your work. Discuss possible causes of error.

(continued)

The example in your book illustrates a method of indirect measurement that involves shadows. Read the example and make sure you can explain why the two triangles are similar. Here is another example.

EXAMPLE A brick building casts a shadow 7 feet long. At the same time, a 3-foot-tall child casts a shadow 6 inches long. How tall is the building?

▶ ***Solution*** The drawing below shows the similar triangles formed. Find x by setting up and solving a proportion.



$$\frac{\text{Height of child}}{\text{Length of child's shadow}} = \frac{\text{Height of building}}{\text{Length of building's shadow}}$$

$$\frac{3}{0.5} = \frac{x}{7}$$

$$6 = \frac{x}{7}$$

$$7 \cdot 6 = x$$

$$42 = x$$

The building is 42 feet tall.

CONDENSED LESSON 11.4

Corresponding Parts of Similar Triangles

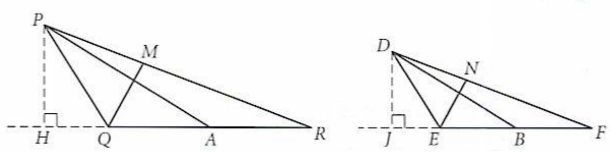
In this lesson you will

- Investigate the relationship between **corresponding altitudes, corresponding medians,** and **corresponding angle bisectors** of similar triangles
- **Prove** that the lengths of corresponding medians of similar triangles are proportional to the lengths of corresponding sides
- Discover a **proportional relationship involving angle bisectors**

If two triangles are similar, then their side lengths are proportional. In the next investigation you will see if there is a relationship between the lengths of corresponding altitudes, corresponding medians, or corresponding angle bisectors.

Investigation 1: Corresponding Parts

$\triangle PQR \sim \triangle DEF$. The scale factor from $\triangle PQR$ to $\triangle DEF$ is $\frac{3}{4}$.



$\overline{PH}$ and $\overline{DJ}$ are corresponding altitudes. How do the lengths of these altitudes compare? How does the comparison relate to the scale factor?

$\overline{PA}$ and $\overline{DB}$ are corresponding medians. How do the lengths of these medians compare?

$\overline{QM}$ and $\overline{EN}$ are corresponding angle bisectors. How do the lengths of these angle bisectors compare?

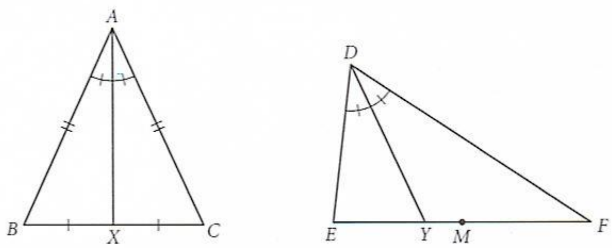
Now draw your own triangle and then construct a similar triangle of a different size. Tell what scale factor you used. Follow Steps 2–4 in your book to construct and compare the lengths of corresponding altitudes, medians, and angle bisectors.

Summarize your findings in this investigation by completing the conjecture below.

Proportional Parts Conjecture If two triangles are similar, then the lengths of the corresponding ________________, ________________, and ________________ are ________________ to the lengths of the corresponding sides. **C-94**

(continued)

If a triangle is isosceles, the bisector of the vertex angle divides the opposite sides into equal parts. (That is, the angle bisector is also a median.) However, as the triangle on the right below shows, this is not true for all triangles.



$\overline{AX}$ is an angle bisector.
Point X is the midpoint of $\overline{BC}$.

$\overline{DY}$ is an angle bisector.
Point M is the midpoint of $\overline{EF}$.

The angle bisector does, however, divide the opposite side in a particular way.

Investigation 2: Opposite Side Ratios

Follow Steps 1–5 in your book.

You should find that both ratios are equal to $\frac{1}{2}$.

Repeat Steps 1–5 in your book with $AC = 6$ units and $AB = 18$ units.

This time you should find that $\frac{CA}{BA}$ and $\frac{CD}{BD}$ are both equal to $\frac{1}{3}$.

You can state your findings as a conjecture.

Angle Bisector/Opposite Side Conjecture A bisector of an angle in a triangle divides the opposite side into two segments whose lengths are in the same ratio as the lengths of the two sides forming the angle. **C-95**

The example in your book proves that in similar triangles, the lengths of the corresponding medians are proportional to the lengths of corresponding sides. Read the example, following along with each step of the proof.

CONDENSED LESSON

11.5 Proportions with Area

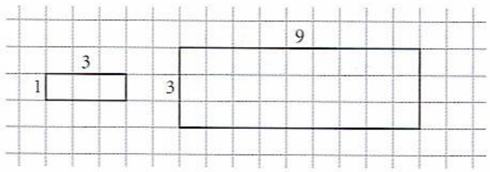
In this lesson you will

- Discover the **relationship between the areas** of similar figures

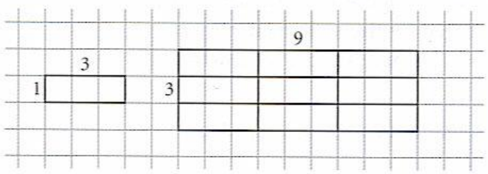
How does multiplying the dimensions of a two-dimensional figure by a scale factor affect its area? In this lesson you will explore this question.

Investigation 1: Area Ratios

The rectangle on the right was created by multiplying the side lengths of the rectangle on the left by 3.



The area of the small rectangle is 3 square units. The area of the large rectangle is 27 square units. The ratio of side lengths of the larger rectangle to side lengths of the smaller rectangle is $\frac{3}{1}$, and the ratio of areas is $\frac{9}{1}$. Notice that nine copies of the small rectangle fit inside the large rectangle.



Now draw your own rectangle on graph paper. Then create a larger or smaller rectangle by multiplying the sides by a scale factor other than 3. What is the ratio of side lengths of the larger rectangle to side lengths of the smaller rectangle? What is the ratio of the areas? How do the two ratios compare?

Draw a triangle on graph paper (your work will be easiest if you draw a right triangle). Draw a similar triangle by multiplying the side lengths by a scale factor. Find the ratio of side lengths and the ratio of areas. How do the ratios compare?

Do you think your findings would be the same for any pair of polygons?

Would your findings be the same for a circle? Consider a circle with radius 5 cm and a circle with radius 20 cm. How does the ratio of radii compare to the ratio of areas?

(continued)

The following conjecture summarizes the relationship between the areas of similar figures.

Proportional Areas Conjecture If corresponding side lengths of two similar polygons or the radii of two circles compare in the ratio $\frac{m}{n}$, then their areas compare in the ratio $\frac{m^2}{n^2}$. **C-96**

The reason behind the Proportional Areas Conjecture is that area is a two-dimensional measure. Calculating area involves multiplying two linear measures, such as base and height. So, if a rectangle has base b and height h, then its area is bh. If the base and height are each multiplied by 2, then the area of the new rectangle is $2b \cdot 2h$, or $4bh$. This is four times the area of the original rectangle. Similarly, if the base and height are each multiplied by 3, then the area of the new rectangle is $3b \cdot 3h$, or $9bh$. This is nine times the area of the original rectangle.

Investigation 2: Surface Area Ratios

In this investigation you will explore whether the Proportional Areas Conjecture is true for surface areas of similar figures. For this investigation you will need interlocking cubes and isometric dot paper.

Follow Steps 1–3 in your book.

You should find that the surface area for the Step 1 figure is 22 square units, while the surface area for a similar prism enlarged by the scale factor 2 is 88 square units. So, lengths of corresponding edges have ratio 2 to 1, while the ratio of the surface areas is 88 to 22, or 4 to 1.

Follow Step 4 in your book. Make sure you include the area of every face, including the face that the figure is resting on.

You should find that the surface area of the figure is 28 square units.

Follow Steps 5 and 6 in your book.

If your answers for Steps 5 and 6 are correct, you can correctly conclude that the Proportional Areas Conjecture also applies to surface area.

The example in your book shows how to use the Proportional Areas Conjecture to solve a real-life problem. Solve the problem yourself before reading the solution.

CONDENSED
LESSON

11.6 Proportions with Volume

In this lesson you will

- Discover the **relationship between the volumes** of similar solids

How does multiplying every dimension of a three-dimensional solid by the same scale factor affect its volume? In this lesson you will explore this question.

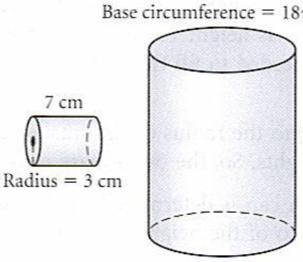
Suppose you are going to create a statue that is 2 feet high. You first create a smaller version of the statue that is 4 inches tall and weighs 8 ounces. How much clay should you buy for the larger statue? Would you believe you'll need 108 pounds of clay? When you finish this lesson, you'll understand why.

Similar solids are solids that have the same shape, but not necessarily the same size. All cubes are similar, but not all prisms are similar. All spheres are similar, but not all cylinders are similar. Two polyhedrons are similar if all their corresponding faces are similar and the lengths of their corresponding edges are proportional. Two right cylinders or right cones are similar if their radii and heights are proportional.

Examples A and B in your book involve determining whether two given solids are similar. Try to answer the problems yourself before reading the solutions.

Here is another example.

EXAMPLE Are these two right cylinders similar?



Solution Find the radius of the larger cylinder.

$$C = 2\pi r$$

$$18\pi = 2\pi r$$

$$r = 9$$

The radius is 9 cm.

Comparing the lengths of corresponding parts we see:

The ratio of the radii is $\frac{3}{9} = \frac{1}{3}$.

The ratio of the heights is $\frac{7}{21} = \frac{1}{3}$.

The radii and heights are proportional, so the right cylinders are similar.

(continued)

Investigation: Volume Ratios

In this investigation you'll explore how the ratio of the edge lengths of similar solids compares to the ratio of the volumes. You will need interlocking cubes and isometric dot paper.

Follow Steps 1 and 2 in your book. In Step 2, make sure you multiply all three dimensions—length, width, and height—by 2.

What is the ratio of the side lengths (larger to smaller) for the two "snakes"? What is the ratio of the volumes? How do the ratios compare?

Follow Steps 4–6 in your book. How would the volume change if you multiplied each dimension by 5? By $\frac{1}{2}$?

Your findings can be stated as a conjecture.

Proportional Volumes Conjecture If corresponding edge lengths (or radii, or heights) of two similar solids compare in the ratio $\frac{m}{n}$, then their volumes compare in the ratio $\frac{m^3}{n^3}$. **C-97**

The example in your book shows how to apply both the Proportional Areas Conjecture and the Proportional Volumes Conjecture. Work through the example to understand how each conjecture is used. Then read the example below. Solve the problem yourself before reading the solution.

EXAMPLE A "square can" is a right cylinder that has a height equal to its diameter. One square can has height 5 cm and another has height 12 cm. About how many full cans of water from the smaller can are needed to fill the larger can?

Solution The ratio of the radius of the larger can to the radius of the smaller can is $\frac{6}{2.5} = \frac{12}{5}$, which is the same as the ratio of the heights. So, the two square cans are similar.

The amount of water needed to fill each can is determined by the volume of the can, so find the ratio of the volumes. The ratio of the heights is $\frac{12}{5}$. Therefore, the ratio of the volumes is $\frac{12^3}{5^3}$, or $\frac{1728}{125}$.

$\frac{1728}{125} \approx 13.8$, so it takes almost 14 cans of water from the smaller can to fill the larger can.

CONDENSED LESSON

11.7 Proportional Segments Between Parallel Lines

In this lesson you will

- Explore the relationships in the lengths of segments formed when one or more lines parallel to one side of a triangle intersect the other two sides
- Learn how you can use the relationship you discover to divide a given segment into any number of equal parts

The top of page 623 in your book shows $\triangle LUV$ and line MT, with $\overleftrightarrow{MT} \parallel \overline{LU}$. It appears that $\triangle LUV \sim \triangle MTV$. The paragraph proof given uses the AA Similarity Conjecture to prove this is true. Example A in your book uses the similarity of two triangles to solve a problem. Read the example, and follow along with each step in the solution.

Look at the figure from Example A. Notice that $\frac{LE}{EM} = \frac{45}{60} = \frac{3}{4}$ and $\frac{NO}{OM} = \frac{36}{48} = \frac{3}{4}$, so there are more relationships in the figure than the ones found by using similar triangles. You will explore these relationships in the next investigation.

Investigation 1: Parallels and Proportionality

Step 1 of the investigation gives three triangles, each with a line parallel to one side that intersects the other two sides. For each triangle, find x and then find the values of the specified ratios. Here is the solution to part a.

a. Use the fact that $\triangle CDE \sim \triangle BDA$ to write and solve a proportion.

$\frac{DE}{DA} = \frac{DC}{DB}$ Corresponding parts of similar triangles are proportional.

$\frac{8}{24} = \frac{12}{12 + x}$ Substitute the lengths from the figure.

$\frac{1}{3} = \frac{12}{12 + x}$ Simplify the left side.

$\frac{1(3)(12 + x)}{3} = \frac{12(3)(12 + x)}{12 + x}$ Multiply both sides by $3(12 + x)$.

$12 + x = 36$ Simplify.

$x = 24$ Subtract 12 from both sides.

So, $\frac{DE}{AE} = \frac{8}{16} = \frac{1}{2}$ and $\frac{DC}{BC} = \frac{12}{24} = \frac{1}{2}$.

In each part of Step 1, you should find that the ratios of the lengths of the segments cut by the parallel line are equal. In other words: If a line parallel to one side of a triangle passes through the other two sides, then it divides the other two sides proportionally.

Do you think the converse of this statement is also true? That is, if a line divides two sides of a triangle proportionally, is it parallel to the third side?

Follow Steps 3–7 in your book. You should find that $\angle PAC \cong \angle PBD$, so $\overline{AC}$ and $\overline{BD}$ are parallel.

Repeat Steps 3–7, but this time mark off your own lengths, such that $\frac{PA}{AB} = \frac{PC}{CD}$. Again, you should find that $\overline{AC}$ is parallel to $\overline{BD}$. You can use your findings from this investigation to state a conjecture.

(continued)

Parallel/Proportionality Conjecture If a line parallel to one side of a triangle passes through the other two sides, then it divides the other two sides proportionally. Conversely, if a line cuts two sides of a triangle proportionally, then it is parallel to the third side. **C-98**

Example B proves the first part of the Parallel/Proportionality Conjecture. Write a proof yourself before reading the one in the book. Use the fact that $\triangle AXY \sim \triangle ABC$ to set up a proportion. Then write a series of algebraic steps until you get $\frac{a}{c} = \frac{b}{d}$.

Investigation 2: Extended Parallel/Proportionality

In the triangles in Step 1, more than one segment is drawn parallel to one side of a triangle. Find the missing lengths. Here is the solution to part a. To find x and y, apply the Parallel/Proportionality Conjecture to the appropriate triangles and lines.

a. To find x, use $\triangle AEL$ and $\overline{FT}$.

$$\frac{EF}{FL} = \frac{ET}{TA}$$

$$\frac{21}{35} = \frac{42}{x}$$

$$x = 70$$

To find y, use $\triangle REG$ and $\overline{LA}$.

$$\frac{EL}{LG} = \frac{EA}{AR}$$

$$\frac{56}{28} = \frac{112}{y}$$

$$y = 56$$

Using the values of x and y, you can see that $\frac{FL}{LG} = \frac{TA}{AR} = \frac{5}{4}$.

The results of Step 1 lead to the following conjecture.

Extended Parallel/Proportionality Conjecture If two or more lines pass through two sides of a triangle parallel to the third side, then they divide the two sides proportionally. **C-99**

You can use the Extended Parallel/Proportionality Conjecture to divide a segment into any number of equal parts. Example C in your book shows you how to divide a segment, AB, into three equal parts. Read the example carefully. To make sure you understand the process, divide $\overline{XY}$ below into three equal parts using a compass and straightedge.